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Wronskian determinants, the KP hierarchy and supersymmetric polynomials

J J C Nimmo

Department of Mathematics, University of Glasgow, Glasgow, UK

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Abstract. By using the Wronskian representation of the solutions of the bilinear KP hierarchy, a connection between Hirota derivatives and supersymmetric polynomials is brought to light. This correspondence is used in order to give an alternative construction of the hierarchy.

1. Introduction

The most widely studied soliton equation in $(2+1)$ dimensions is the Kadomtsev-Petviashvili (KP) equation (Kadomtsev and Petviashvili 1970). These investigations include the inverse scattering transform in the plane (for example Manakov 1981, Ablowitz *et al* 1983) and 'bilocal' recursion operators (Santini and Fokas 1986) as well as applications of Hirota's direct method for obtaining soliton and lump solutions (Satsuma 1976, Satsuma and Ablowitz 1979).

The τ function approach (Sato 1981, Jimbo and Miwa 1983) has also been important in bringing to light the algebraic properties of the KP equation. In this theory one may obtain a hierarchy of Hirota equations in infinitely many independent variables satisfied by the same solutions, and called the (bilinear) KP hierarchy, of which the KP equation is the base member. The aim of this paper is to show how this hierarchy may be constructed directly, using techniques from symmetric function theory. Here we prove a slightly modified version of an earlier conjecture (Nimmo 1988a).

Underlying this construction is the representation of solutions of the KP hierarchy as Wronskian determinants (Freeman and Nimmo 1983). It will turn out that, if one considers solutions represented in this way, then derivatives correspond to certain power-sum *symmetric* functions and Hirota derivatives to power-sum *supersymmetric* functions. This relationship is the key to the construction.

The construction may also be cast in terms of an infinite family of operators bearing a tantalising resemblance to the bilocal recursion operators of the more usual—evolution equation—form of the KP hierarchy. It has not, however, been possible to make this connection at all concrete.

2. The KP equation

The KP equation

$$(u_t+6uu_x+u_{xxx})_x+3u_{yy}=0 \tag{1}$$

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is transformed to the Hirota form by the change of variable

$$u=2\frac{\partial^2}{\partial x^2}\log\tau \tag{2}$$

and we have

$$(D_x^4+D_xD_t+3D_y^2)\tau\cdot\tau=0 \tag{3}$$

where the Hirota derivatives are defined by

$$D_x^mD_t^n\sigma\cdot\tau\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n\sigma(x,t)\tau(x',t')\bigg|_{\substack{x'=x\\ t'=t}}.$$

In order to achieve a uniform notation we introduce an infinite sequence of independent variables x_k, $k\in\mathbb{N}$, in which we have $x_1=x$, $x_2=y$ and $x_3=-\frac{1}{4}t$. Furthermore, we introduce partition notation for derivatives and Hirota derivatives;

$$\partial_\lambda\equiv\frac{\partial^p}{\partial x_{\lambda_1}\ldots\partial x_{\lambda_p}}$$

$$D_\lambda\equiv D_{x_{\lambda_1}}\ldots D_{x_{\lambda_p}}$$

for any partition $\lambda=(\lambda_1,\ldots,\lambda_p)$. We say that a Hirota derivative D_λ is of weight k if λ is a partition of the integer k.

Throughout this paper we will use the notation and ideas of symmetric functions. For readers not familiar with this theory the book by Macdonald (1979) is particularly recommended. In this notation the KP equation has the Hirota form

$$(D_{(1^4)}+3D_{(2^2)}-4D_{(31)})\tau\cdot\tau=0 \tag{4}$$

where τ is now taken to be a function of the sequence of variables $\boldsymbol{x}=(x_1,x_2,x_3,\ldots)$. One may show that the Wronskian determinant

$$\tau=W(\varphi_1,\ldots,\varphi_N)=\det(\partial^{i-1}\varphi_j) \tag{5}$$

with $\partial\equiv\partial/\partial x_1$, satisfies (4) provided, for $j=1,\ldots,N$, $\varphi_j(\boldsymbol{x})$ satisfy

$$\frac{\partial^k\varphi}{\partial x_1^k}=\frac{\partial\varphi}{\partial x_k}\qquad(k\in\mathbb{N}) \tag{6}$$

(Freeman and Nimmo 1983). This result is achieved by observing that (4), with τ as in (5), is the expansion by $N\times N$ minors of the determinant

$$\left|\begin{array}{c:c}\boldsymbol{\varphi}^{(0)},\ldots,\boldsymbol{\varphi}^{(N-3)} & \boldsymbol{\varphi}^{(0)},\ldots,\boldsymbol{\varphi}^{(N+1)}\\ \hdashline \mathbf{0} & \boldsymbol{\varphi}^{(0)},\ldots,\boldsymbol{\varphi}^{(N+1)}\end{array}\right|=0 \tag{7}$$

where $\boldsymbol{\varphi}^{(i)}\equiv(\partial^i/\partial x_1^i)(\varphi_1,\ldots,\varphi_N)^T$ and $\mathbf{0}$ the zero matrix of appropriate size.

We wish to identify all Hirota equations

$$\left(\sum_\lambda a_\lambda D_\lambda\right)\tau\cdot\tau=0 \tag{8}$$

for constants a_λ, that are satisfied by the Wronskian (5) for all N. These equations will constitute the KP hierarchy. This question has been addressed before by Sato (1981) where it was shown that these equations correspond to Plücker relations on an infinite-dimensional Grassmann manifold, of which (7) is an example. The lowest-weight equations are listed in Jimbo and Miwa (1983). The intention here is to describe another construction of this hierarchy using only properties of the solution (5). We observe that, if a Hirota equation of the form (8) is satisfied by a Wronskian $\tau^{(N)}$ of any N functions $\varphi_1(\boldsymbol{x}),\ldots,\varphi_N(\boldsymbol{x})$ satisfying (6), then it is satisfied by the Wronskian $\tau^{(N-1)}$ of the $(N-1)$ functions $\partial\varphi_2(\boldsymbol{x}),\ldots,\partial\varphi_N(\boldsymbol{x})$, since if we take $\varphi_1(\boldsymbol{x})=1$

$$\tau^{(N)}=\begin{vmatrix} 1 & 0 & \cdots & 0 \\ \varphi_2(\boldsymbol{x}) & \partial\varphi_2(\boldsymbol{x}) & \cdots & \partial^{N-1}\varphi_2(\boldsymbol{x}) \\ \vdots & \vdots & & \vdots \\ \varphi_N(\boldsymbol{x}) & \varphi_N(\boldsymbol{x}) & \cdots & \partial^{N-1}\varphi_N(\boldsymbol{x}) \end{vmatrix} = W(\partial\varphi_2(\boldsymbol{x}),\ldots,\partial\varphi_N(\boldsymbol{x}))=\tau^{(N-1)}.$$

Hence, to identify members of the KP hierarchy, we may assume that $N=K$ is arbitrarily large, find those Hirota equations satisfied by $\tau^{(K)}$ and then use the above observation to deduce that these equations must be satisfied for all $N\leqslant K$.

The hierarchy contains all Hirota equations of the form (8) in which a_λ is non-zero only if λ is a partition with an odd number of parts. The skew-symmetric nature of Hirota derivatives means that such a Hirota polynomial is satisfied by *any* function τ. Such odd Hirota equations are hence trivial. The first non-trivial member of the hierarchy is the KP equation (4). We shall exhibit two constructions: one generating all equations and the other only the non-trivial ones.

3. Supersymmetric polynomials and the KP hierarchy

First of all we will show how derivatives and Hirota derivatives correspond to certain symmetric functions when acting on the Wronskian determinant (5). Rewrite (5) as

$$\tau = V(\partial_1,\ldots,\partial_N)\prod_{i=1}^{N}\varphi_i(\boldsymbol{x}_i)\Big|_{\boldsymbol{x}_1=\ldots=\boldsymbol{x}_N=\boldsymbol{x}} \tag{9}$$

where we introduce a copy, $\boldsymbol{x}_i$, of the infinite sequence of independent variables $\boldsymbol{x}$ for each function φ_i, and let ∂_i denote the corresponding copy of ∂. In (9) $V(\partial_1,\ldots,\partial_N)$ denotes the Vandermonde determinant of its arguments. Since the φ_i satisfy (6), it may be shown that

$$\partial_\lambda\tau = p_\lambda(\partial_1,\ldots,\partial_N)V(\partial_1,\ldots,\partial_N)\prod_{i=1}^{N}\varphi_i(\boldsymbol{x}_i)\Big|_{\boldsymbol{x}_1=\ldots=\boldsymbol{x}_N=\boldsymbol{x}} \tag{10}$$

where

$$p_\lambda(\partial_1,\ldots,\partial_N)=\prod_{i=1}^{p}\left(\sum_{j=1}^{N}\partial_j\lambda_i\right) \tag{11}$$

is the power-sum symmetric function for the partition $\lambda=(\lambda_1,\ldots,\lambda_p)$. An obvious extension of this idea shows that

$$D_\lambda\tau\cdot\tau=\tilde{p}_\lambda(\partial_1,\ldots,\partial_N;\bar{\partial}_1,\ldots,\bar{\partial}_N)V(\partial_1,\ldots,\partial_N)V(\bar{\partial}_1,\ldots,\bar{\partial}_N)$$
$$\times\prod_{i=1}^{N}\varphi_i(\boldsymbol{x}_i)\varphi_i(\bar{\boldsymbol{x}}_i)\Big|_{\boldsymbol{x}_1=\ldots=\boldsymbol{x}_N=\bar{\boldsymbol{x}}_1=\ldots=\bar{\boldsymbol{x}}_N=\boldsymbol{x}} \tag{12}$$

where

$$\tilde{p}_\lambda(\partial_1,\ldots,\partial_N;\bar{\partial}_1,\ldots,\bar{\partial}_N)\equiv\prod_{i=1}^{p}(p_{\lambda_i}(\partial_1,\ldots,\partial_N)-p_{\lambda_i}(\bar{\partial}_1,\ldots,\bar{\partial}_N))$$

for the partition $\lambda=(\lambda_1,\ldots,\lambda_p)$. The polynomials $\tilde{p}_\lambda$ were first introduced by Littlewood (1950) and called generalised power-sum symmetric functions. More recently they have been used in the representation theory of supergroups where they are called power-sum supersymmetric functions (see, for example, King 1983). Now we shall consider the properties of such polynomials in some detail before returning to the current consideration.

A polynomial $f(\boldsymbol{u};\boldsymbol{v})$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are sets of independent variables $u_1,\ldots,u_N$ and $v_1,\ldots,v_M$ is said to be *doubly symmetric* if f is invariant under permutations of $\boldsymbol{u}$ and $\boldsymbol{v}$. Further, a doubly symmetric polynomial $f(\boldsymbol{u};\boldsymbol{v})$ is said to be *supersymmetric* if, for any $i=1,\ldots,N$ and $j=1,\ldots,M$,

$$f(\boldsymbol{u};\boldsymbol{v})|_{u_i=v_j=z} \qquad \text{is independent of } z. \tag{13}$$

It has been shown (Scheunert 1982, Stembridge 1985) that a polynomial $f(\boldsymbol{u};\boldsymbol{v})$ is supersymmetric iff

$$f(\boldsymbol{u};\boldsymbol{v})=\sum_\lambda c_\lambda\tilde{p}_\lambda(\boldsymbol{u};\boldsymbol{v}) \tag{14}$$

for some constants c_λ, where the sum is over all partitions. We shall use this result in order to utilise the criterion (13) as a means of testing whether a polynomial is a linear combination of power-sum supersymmetric functions. We shall only be interested in the case $N=M$ here.

Define the $2N\times 2N$ determinants:

$$\Delta^m_{\lambda,\mu}(\boldsymbol{u};\boldsymbol{v})=\left|\begin{array}{c:c} M^{N-m}_\lambda(\boldsymbol{u}) & M^{N+m}_\mu(\boldsymbol{u}) \\ \hdashline \mathbf{0} & M^{N+m}_\mu(\boldsymbol{v}) \end{array}\right| \tag{15}$$

where

$$M^k_\rho(\boldsymbol{z})=\begin{pmatrix} 1 & z_1 & \cdots & z_1^{k-p-1} & z_1^{\rho_p+k-p} & \cdots & z_1^{\rho_1+k-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & z_N & \cdots & z_N^{k-p-1} & z_N^{\rho_p+k-p} & \cdots & z_N^{\rho_1+k-1} \end{pmatrix}$$

is an $N\times k$ matrix, ρ is the partition $(\rho_1,\ldots,\rho_p)$ and $0\leqslant m\leqslant N$. Here, and in all that follows, we shall assume that N is sufficiently large for definitions to be meaningful and labellings of columns of determinants to make sense; here, for example, we require that N is such that $N-m>l(\lambda)$ and $N+m>l(\mu)$, i.e. $N>\max\{l(\lambda)+m, l(\mu)-m\}$, where $l(\lambda)$ is the number of parts in the partition λ.

We use the above determinants to define a family of doubly symmetric polynomials:

$$S^m_{\lambda,\mu}(\boldsymbol{u};\boldsymbol{v})=\frac{\Delta^m_{\lambda,\mu}(\boldsymbol{u};\boldsymbol{v})}{V(\boldsymbol{u})V(\boldsymbol{v})} \tag{16}$$

where $V(\boldsymbol{z})=\prod_{i>j}(z_i-z_j)$ is the Vandermonde determinant. This definition is very similar to the usual representation of a Schur function as the ratio of determinants and it is important to note that the ratio in (16) *is* a polynomial since the numerator vanishes when any pair of the u_i or v_i are identified, and hence has $V(\boldsymbol{u})V(\boldsymbol{v})$ as a factor. In general these doubly symmetric polynomials are not supersymmetric but one may prove the following result (see the appendix).

Theorem. For sufficiently large N, the doubly symmetric polynomials

$$E^m_\lambda(\boldsymbol{u};\boldsymbol{v})=\sum_{\mu,\rho}(-1)^{|\mu|}c^\lambda_{\mu\rho}S^m_{\rho,\mu'}(\boldsymbol{u};\boldsymbol{v}) \qquad (m\geqslant 0) \tag{17}$$

where the $c^{\lambda}_{\mu\rho}$ are constants defined in (A4), are supersymmetric. Hence for some constants a_μ, depending on m and λ,

$$E^m_\lambda(\boldsymbol{u};\boldsymbol{v})=\sum_\mu a_\mu\tilde{p}_\mu(\boldsymbol{u};\boldsymbol{v}). \tag{18}$$

It remains to interpret this result in the context of Hirota derivatives and the KP hierarchy. Consider the determinants

$$\mathfrak{T}^m_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{y})=\left|\begin{array}{c:c}\mathfrak{M}^{N-m}_\lambda(\boldsymbol{x}) & \mathfrak{M}^{N+m}_\mu(\boldsymbol{y})\\ \hdashline \mathbf{0} & \mathfrak{M}^{N+m}_\mu(\boldsymbol{y})\end{array}\right|, \tag{19}$$

where

$$\mathfrak{M}^k_\rho(\boldsymbol{z})=\begin{pmatrix}\varphi_1 & \partial\varphi_1 & \dots & \partial^{\rho_p+k-p}\varphi_1 & \dots & \partial^{\rho_1+k-1}\varphi_1\\ \vdots & \vdots & & \vdots & & \vdots\\ \varphi_N & \partial\varphi_N & \dots & \partial^{\rho_p+k-p}\varphi_N & \dots & \partial^{\rho_1+k-1}\varphi_N\end{pmatrix}$$

(cf (15)). In (7) we have $\mathfrak{T}^2_{(\cdot),(\cdot)}=0$. Observe that

$$(M^k_\rho(\partial_1,\dots,\partial_N))^T\operatorname{diag}(\varphi_1(\boldsymbol{x}_1),\dots,\varphi_N(\boldsymbol{x}_N))|_{\boldsymbol{x}_1=\dots=\boldsymbol{x}_N=\boldsymbol{x}}=(\mathfrak{M}^k_\rho(\boldsymbol{x}))^T$$

where M^k_ρ is as defined in (15), and so

$$\begin{aligned}\mathfrak{T}^m_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{x})&=\Delta^m_{\lambda,\mu}(\partial_1,\dots,\partial_N;\bar{\partial}_1,\dots,\bar{\partial}_N)\prod_{i=1}^N\varphi_i(\boldsymbol{x}_i)\varphi_i(\bar{\boldsymbol{x}}_i)\Big|_{\boldsymbol{x}_1=\dots=\boldsymbol{x}_N=\bar{\boldsymbol{x}}_1=\dots=\bar{\boldsymbol{x}}_N=\boldsymbol{x}}\\&=S_{\lambda,\mu}(\partial_1,\dots,\partial_N;\bar{\partial}_1,\dots,\bar{\partial}_N)V(\partial_1,\dots,\partial_N)V(\bar{\partial}_1,\dots,\bar{\partial}_N)\\&\quad\times\prod_{i=1}^N\varphi_i(\boldsymbol{x}_i)\varphi_i(\bar{\boldsymbol{x}}_i)\Big|_{\boldsymbol{x}_1=\dots=\boldsymbol{x}_N=\bar{\boldsymbol{x}}_1=\dots=\bar{\boldsymbol{x}}_N=\boldsymbol{x}}\end{aligned}$$

This leads to a corollary to the theorem:

$$\mathfrak{H}^m_\lambda\equiv\sum_{\mu,\rho}(-1)^{|\mu|}c^\lambda_{\mu\rho}\mathfrak{T}^m_{\rho,\mu'}(\boldsymbol{x};\boldsymbol{x})=\left(\sum_\mu a_\mu D_\mu\right)\tau\cdot\tau \tag{20}$$

when τ is the Wronskian given in (5). For $m>0$, the determinants $\mathfrak{T}^m_{\mu,\rho}$ are zero and so the $\mathfrak{H}^m_\lambda$, for any partition λ and any $m>0$, are Hirota equations satisfied by the Wronskian (5).

It is possible to restate this result in terms of a family of linear operators $\mathfrak{R}_k$ ($k\in\mathbb{N}$), defined by

$$\mathfrak{R}_k(\mathfrak{T}^m_{\mu,\rho}(\boldsymbol{x};\boldsymbol{x}))=\left(\frac{\partial}{\partial x_k}-\frac{\partial}{\partial y_k}\right)\mathfrak{T}^m_{\mu,\rho}(\boldsymbol{x};\boldsymbol{y})\Big|_{\boldsymbol{y}=\boldsymbol{x}} \tag{21}$$

which is reminiscent of the definition of Hirota derivatives. In fact, for $m=0$ and $\mu=\rho=(\cdot)$, this coincides precisely with the definition of D_{x_k} since

$$\mathfrak{T}^0_{(\cdot),(\cdot)}(\boldsymbol{x};\boldsymbol{y})=\tau(\boldsymbol{x})\tau(\boldsymbol{y}).$$

We also define $\mathfrak{R}_\lambda$, for any partition λ, in an obvious way. By using the relationship between power sums and Schur functions (cf (A2)) we have, for any partition μ and $m>0$,

$$\mathfrak{R}_\mu(\mathfrak{H}^m_{(\cdot)})=\mathfrak{R}_\mu(\mathfrak{T}^m_{(\cdot),(\cdot)}(\boldsymbol{x};\boldsymbol{x}))=\sum\chi^\lambda_\mu\mathfrak{H}^m_\lambda=0 \tag{22}$$

which is a Hirota equation satisfied by the Wronskian τ.

Since $\mathfrak{T}^m_{(\cdot),(\cdot)}(\boldsymbol{x};\boldsymbol{x})$ ($m>0$) is a Hirota equation of homogeneous weight m^2, this construction generates $P(k-m^2)$ Hirota equations of weight k, where $P(n)$ is the number of partitions of the integer n. We use the convention $P(n)=0$ for $n<0$.

For $m = 1$, this corresponds exactly with the result of Sato (1981) and the construction gives the whole KP hierarchy including the trivial, odd, Hirota equations such as the base member

$$\mathfrak{H}^1_{(\cdot)} = D_{(1)}\tau \cdot \tau = 0.$$

If one takes $m = 2$, however, the base member is the weight-4 KP equation (4), in the current notation

$$\mathfrak{H}^2_{(\cdot)} = \tfrac{1}{6}(D_{(1^4)} + 3D_{(2^2)} - 4D_{(31)})\tau \cdot \tau = 0.$$

The next members of the hierarchy are, at weight 5,

$$\mathfrak{R}_{(1)}(\mathfrak{H}^2_{(\cdot)}) = \mathfrak{H}^2_{(1)} = \tfrac{1}{3}(D_{(21^3)} + 2D_{(32)} - 3D_{(41)})\tau \cdot \tau = 0$$

and, at weight 6,

$$\mathfrak{R}_{(2)}(\mathfrak{H}^2_{(\cdot)}) = \mathfrak{H}^2_{(2)} - \mathfrak{H}^2_{(1^2)} = \tfrac{1}{90}(D_{(1^6)} + 10D_{(31^3)} - 20D_{(3^2)} + 45D_{(42)} - 36D_{(51)})$$

$$\mathfrak{R}_{(1^2)}(\mathfrak{H}^2_{(\cdot)}) = \mathfrak{H}^2_{(2)} + \mathfrak{H}^2_{(1^2)} = -\tfrac{1}{180}(D_{(1^6)} - 45D_{(2^21^2)} - 20D_{(31^3)} - 80D_{(3^2)} + 144D_{(51)}).$$

The number of equations of weight k generated in this case is $P(k-4)$ and for weights 4–8 inclusive these equations are precisely the non-trivial, even order, members of the hierarchy. For $k \geqslant 9$ the construction gives equations with terms of both odd and even order. These equations, although themselves linearly independent, become linearly dependent when one eliminates the trivial terms.

In fact, the number of independent non-trivial equations of weight k is $P(k-1) - P_0(k)$, where $P_0(n)$ is the number of partitions of n into an odd number of parts. This is consistent with the fact that, for $1 < k < 9$, $P(k-1) - P_0(k) = P(k-4)$, but for $k \geqslant 9$ $P(k-1) - P_0(k) < P(k-4)$. More details and examples of the construction of this non-trivial hierarchy are given in Nimmo (1988a).

4. Conclusions

We have given an alternative construction of the KP hierarchy using symmetric function techniques. As a consequence of this we have brought to light a connection between Hirota derivatives and power-sum supersymmetric functions. It would be interesting to see if any of the theory of the KP and other hierarchies could be utilised in the representation theory where these functions arise.

This type of construction through the recursion-like operators $\mathfrak{R}_k$ may be carried out in other settings. We have already considered elsewhere (Nimmo 1988a) two other hierarchies described in Jimbo and Miwa (1983); the modified KP hierarchies where the construction is identical—the proofs given here need little modification—and the BKP hierarchy where the solutions are not Wronskians but the same type of construction appears to work.

The higher members of the KP hierarchy have no direct relevance to the study of physically interesting non-linear evolution equations. In the multicomponent KP hierarchies, however, this is not so. In these cases solutions take the form of multi-component Wronskians (Nimmo 1988b) and results analagous to those given here may be used to obtain the Hirota equation satisfied by these functions. The equations of dispersive water waves and their modifications (Kuperschmidt 1985, Antonowicz and Fordy 1988) have been studied as an example of this technique (Freeman *et al* 1988).

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Appendix

Consider the supersymmetric Schur functions defined by

$$\tilde{S}_\mu(\boldsymbol{u};\,\boldsymbol{v})=\sum_\lambda z_\lambda^{-1}\chi_\lambda^\mu\tilde{p}_\lambda(\boldsymbol{u};\,\boldsymbol{v}) \tag{A1}$$

where χ_λ^μ are group characters of the symmetric group and z_λ are such that

$$\tilde{p}_\lambda(\boldsymbol{u};\,\boldsymbol{v})=\sum_\mu \chi_\lambda^\mu\tilde{S}_\mu(\boldsymbol{u};\,\boldsymbol{v}) \tag{A2}$$

(cf the relationship between the usual Schur functions and power-sum symmetric functions). Such functions may be expressed as quadratics in ordinary Schur functions using a result due to Littlewood (1950, p 115):

$$\tilde{S}_\lambda(\boldsymbol{u};\,\boldsymbol{v})=\sum_{\mu,\rho}(-1)^{|\mu|}c_{\mu\rho}^\lambda S_\rho(\boldsymbol{u})S_{\mu'}(\boldsymbol{v}) \tag{A3}$$

where the $c_{\mu\rho}^\lambda$ are the constants obtained using the Littlewood–Richardson rule in the product of Schur functions:

$$S_\mu S_\rho=\sum_\lambda c_{\mu\rho}^\lambda S_\lambda. \tag{A4}$$

Also

$$\Delta_{\lambda,\mu}^0(\boldsymbol{u};\,\boldsymbol{v})=\det(M_\lambda^0(\boldsymbol{u}))\cdot\det(M_\mu^0(\boldsymbol{v}))$$

and so

$$\tilde{S}_\lambda(\boldsymbol{u};\,\boldsymbol{v})=\sum_{\mu,\rho}(-1)^{|\mu|}c_{\mu\rho}^\lambda S_{\rho,\mu'}^0(\boldsymbol{u};\,\boldsymbol{v}). \tag{A5}$$

We now investigate the expansion of the determinants $\Delta_{\lambda,\mu}^m$. The following notation is used; if $\boldsymbol{u}$ denotes the set $u_1,\ldots,u_N$ then let $\boldsymbol{u}'$ denote $u_1,\ldots,u_{N-1}$. Similarly, if M denotes an $N\times k$ matrix M' denotes the $(N-1)\times k$ matrix obtained by omitting the last row. Using this notation, we say that $f(\boldsymbol{u};\,\boldsymbol{v})$ is supersymmetric iff $f(\boldsymbol{u}',z;\,\boldsymbol{v}',z)$ is z independent:

$$\Delta_{\lambda,\mu}^m(\boldsymbol{u}',z;\,\boldsymbol{v}',z)$$

$$=\left|\begin{array}{c|c} M_\lambda'^{N-m}(\boldsymbol{u}') & M_\mu'^{N+m}(\boldsymbol{u}') \\ \hline 1\ldots z^{\lambda_p+N-m-p}\ldots z^{\lambda_1+N-m-1} & 1\ldots z^{\mu_q+N+m-q}\ldots z^{\mu_1+N+m-1} \\ \hline \mathbf{0} & M_\mu'^{N+m}(\boldsymbol{y}') \\ \hline 0\cdots\cdots 0 & 1\ldots z^{\mu_q+N+m-q}\ldots z^{\mu_1+N+m-1} \end{array}\right|$$

$$=z^{N-m-1}\cdot z^{N+m-1}\left|\begin{array}{c|c} M_\lambda'^{N-m}(\boldsymbol{u}') & M_\mu'^{N+m}(\boldsymbol{u}') \\ \hline \ldots z^{-\mu_1+1}\ldots z^{\lambda_p-p+1}\ldots z^{\lambda_1} & 0 \qquad\qquad 0 \\ \hline 0 & M_\mu'^{N+m}(\boldsymbol{v}') \\ \hline 0 \qquad\qquad 0 & \ldots z^{-\lambda_1+1}\ldots z^{\mu_q-q+1}\ldots z^{\mu_1} \end{array}\right|$$

and expanding this determinant by its Nth, and then $2N$th, rows gives

$$\Delta^m_{\lambda,\mu}(\boldsymbol{u}', z; \boldsymbol{v}', z) = z^{2N-2}\left(\sum_{k\geq 1}\left(\sum_{\rho,\upsilon}\beta^k_{\rho,\upsilon}\Delta^m_{\rho,\upsilon}(\boldsymbol{u}'; \boldsymbol{v}')\right)z^k + \mathrm{O}(1)\right) \tag{A6}$$

for some constants $\beta^k_{\rho,\upsilon}$, where the crucial property of this expansion is that these constants *do not* depend on the parameter m which governs the partitioning of $\Delta^m_{\lambda,\mu}$. Once again we have assumed here that N is sufficiently large that the above labelling of powers of z is meaningful. Dividing both sides of (A6) by $V(\boldsymbol{u}', z)V(\boldsymbol{v}', z)$ we get

$$S^m_{\lambda,\mu}(\boldsymbol{u}', z; \boldsymbol{v}', z) = \sum_{k\geq 1}\left(\sum_{\rho,\upsilon}\beta^k_{\rho,\upsilon}S^m_{\rho,\upsilon}(\boldsymbol{u}'; \boldsymbol{v}')\right)z^k + \mathrm{O}(1) \tag{A7}$$

and we see that we will be able to deduce that $S^m_{\lambda,\mu}(\boldsymbol{u}; \boldsymbol{v})$ is supersymmetric if all of the constants $\beta^k_{\rho,\upsilon}$ vanish. This follows because the O(1) term in (A7) is z independent since the left-hand side is a polynomial, according to the discussion following (16), and $V(\boldsymbol{u}', z)V(\boldsymbol{v}', z)$ is of order $2N-2$ in z. To prove the theorem in the main text we must show that

$$E^m_\lambda(\boldsymbol{u}; \boldsymbol{v}) = \sum_{\mu,\rho}(-1)^{|\mu|}c^\lambda_{\mu\rho}S^m_{\mu,\rho}(\boldsymbol{u}; \boldsymbol{v}) \qquad (m \geq 0) \tag{A8}$$

(cf (A5)) is supersymmetric. Using (A7) we have the expansion of the form

$$E^m_\lambda(\boldsymbol{u}', z; \boldsymbol{v}', z) = \sum_{k\geq 1}\left(\sum_{\rho,\upsilon}\alpha^k_{\rho,\upsilon}S^m_{\rho,\upsilon}(\boldsymbol{u}'; \boldsymbol{v}')\right)z^k + \mathrm{O}(1) \qquad (m > 0)$$

and

$$\tilde{S}_\lambda(\boldsymbol{u}', z; \boldsymbol{v}', z) = \sum_{k\geq 1}\left(\sum_{\rho,\upsilon}\alpha^k_{\rho,\upsilon}S^0_{\rho,\upsilon}(\boldsymbol{u}'; \boldsymbol{v}')\right)z^k + \mathrm{O}(1)$$

where, as we described earlier, the constants $\alpha^k_{\rho,\upsilon}$ are the same in both expressions. $\tilde{S}_\lambda(\boldsymbol{u}; \boldsymbol{v})$ is supersymmetric and so $\tilde{S}_\lambda(\boldsymbol{u}', z; \boldsymbol{v}', z)$ must be independent of z. Hence all of the $\alpha^k_{\rho,\upsilon}$ must be zero. Thus $E^m_\lambda(\boldsymbol{u}', z; \boldsymbol{v}', z)$ is independent of z and so $E^m_\lambda(\boldsymbol{u}; \boldsymbol{v})$ is supersymmetric. Thus the theorem is proved.

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